

# REMARKS ON RULED SURFACES AND RANK TWO BUNDLES WITH CANONICAL DETERMINANT AND 4 SECTIONS.

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**ABSTRACT.** Let  $C$  be a smooth irreducible complex projective curve of genus  $g$  and let  $B^k(2, K_C)$  be the Brill-Noether locus parametrizing classes of (semi)-stable vector bundles  $E$  of rank two with canonical determinant over  $C$  with  $h^0(C, E) \geq k$ . We show that  $B^4(2, K_C)$  has an irreducible component  $\mathcal{B}$  of dimension  $3g - 13$  on a general curve  $C$  of genus  $g \geq 8$ . Moreover, we show that for the general element  $[E]$  of  $\mathcal{B}$ ,  $E$  fits into an exact sequence  $0 \rightarrow \mathcal{O}_C(D) \rightarrow E \rightarrow K_C(-D) \rightarrow 0$  with  $D$  a general effective divisor of degree three, and the corresponding coboundary map  $\partial : H^0(C, K_C(-D)) \rightarrow H^1(C, \mathcal{O}_C(D))$  has cokernel of dimension three.

## 1. INTRODUCTION AND STATEMENT OF THE RESULT.

Let  $C$  be a smooth irreducible complex projective curve of genus  $g$  and let  $U_C(r, d)$  be the moduli space of (semi)stable vector bundles of rank  $r$  and degree  $d$  on  $C$ . Inside  $U_C(r, d)$ , consider the Brill-Noether locus  $B^k(r, d)$  parametrizing classes of vector bundles  $[E] \in U_C(r, d)$  having at least  $k$  linearly independent sections. Because of the interest geometry behind such loci, higher rank Brill-Noether theory on algebraic curves has been extensively studied and it is actually an active research area in Algebraic Geometry; see e.g. ([9]), for an overview of some main results, and references below, for more recent ones. Despite these facts, several basic questions concerning e.g. non-emptiness, dimensionality, irreducibility, etc., are still open. Let  $B(2, K_C) \subset U_C(2, 2g - 2)$  be the scheme which parametrizes classes of (semi)stable rank-two vector bundles  $E$  with  $\det(E) = \wedge^2 E = K_C$ ; this scheme is defined as the fiber at  $K_C$  of the determinant map  $\wedge^2 : U_C(2, 2g - 2) \rightarrow \text{Pic}^{2g-2}(C)$ ,  $E \mapsto \wedge^2 E$ .  $B(2, K_C)$  is smooth and irreducible of dimension  $3g - 3$ . Inside  $B(2, K_C)$ , for any integer  $k \geq 0$  define the Brill-Noether locus  $B^k(2, K_C) := \{[E] \in B(2, K_C) : h^0(C, E) \geq k\}$ . These loci have a scheme structure as suitable degeneracy loci and it is known that the expected dimension of  $B^k(2, K_C)$  is given by the Brill-Noether number  $\rho_{K_C}(2, k, g) := 3g - 3 - \binom{k+1}{2}$  (see [3], [12]). For a description of  $B^k(2, K_C)$  for low genus we refer to e.g. ([3], [13], [14]). For  $[E] \in B^k(2, K_C)$ , the infinitesimal behavior of  $B^k(2, K_C)$  at the point  $[E]$  is governed by the symmetric Petri map

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$P_E : \text{Sym}^2(H^0(C, E)) \rightarrow H^0(C, \text{Sym}^2(E))$ , that is, the tangent space to  $B^k(2, K_C)$  at  $[E]$  is identified with the orthogonal to the image of  $P_E$ . In ([19]) it has been proved the injectivity of the symmetric Petri map  $P_E$  on a general curve of genus  $g \geq 1$ , when  $B^k(2, K_C)$  is assumed to be non-empty. A different approach for the injectivity of the symmetric Petri map for  $g \leq 9$  and  $k < 7$  is given in ([2]). The injectivity of the symmetric Petri map implies that on a general curve  $C$ , components of the right dimension in  $B^k(2, K_C)$  are smooth.

A general result on non-emptiness and existence of components in  $B^k(2, K_C)$  of the right dimension on a general curve of genus sufficiently large is given in ([18]), where the proof uses the theory of limit linear series for higher rank. For generalizations of this result and others results about non-emptiness, irreducibility and smoothness, we refer to e.g ([5], [11], [15], [16], [20]). Focusing on ([5]), as a by-product of the more general approach developed therein, the authors prove in particular the existence of irreducible components of  $B^k(2, K_C)$  of the right dimension, for  $C$  a general curve of genus  $g > 3$  and for  $k \leq 3$ ; this has been done by studying certain determinantal loci inside the space of extensions  $\text{Ext}^1(K_C(-D_{k-1}), \mathcal{O}_C(D_{k-1}))$ , for a general effective divisor  $D_{k-1}$  of degree  $k-1 \leq 2$ . For a general curve of genus  $g \geq 5$ , the existence of irreducible components in  $B^4(2, K_C)$  of the right dimension is a particular case of Theorem 1.1 given in ([18]). We follow the approach as in ([5]) to show that there exists an irreducible component  $\mathcal{B} \subseteq B^4(2, K_C)$  of (expected) dimension  $\rho_{K_C}(2, 4, g)$  on a general curve of genus  $g \geq 8$  moreover we show that the general vector bundle in  $\mathcal{B}$  is given as an extension of suitable line bundles as in ([5]), to which the reader is referred for more details. In order to state our main theorem we recall here some definitions and notation.

Let  $C$  be a non-hyperelliptic curve of genus  $g \geq 3$  and let  $d$  be an integer such that  $2g-2 \leq d \leq 4g-4$ . Let  $\delta \leq d$  be a positive integer. Let  $L \in \text{Pic}^\delta(C)$ ,  $N \in \text{Pic}^{d-\delta}(C)$  be line bundles on  $C$ ; the space  $\text{Ext}^1(L, N)$  parametrizes isomorphism classes of extensions

$$0 \rightarrow N \rightarrow E \rightarrow L \rightarrow 0.$$

Any  $u \in \text{Ext}^1(L, N)$  gives rise to a degree  $d$  rank-two vector bundle  $E_u$  that fits in an exact sequence

$$(u) : \quad 0 \rightarrow N \rightarrow E_u \rightarrow L \rightarrow 0.$$

To get a (semi)stable vector bundle, a necessary condition is  $2\delta - d \geq 0$  (cf. [5], Remark 2.5 and Section 4.1). By Riemann-Roch,  $\text{Ext}^1(L, N) \simeq H^1(C, N \otimes L^\vee) \simeq H^0(C, K_C \otimes L \otimes N^\vee)^\vee$ , then  $m := \dim(\text{Ext}^1(L, N)) = 2\delta - d + g - 1$  if  $L \not\simeq N$ , and  $m = g$  when  $L \simeq N$ .

Suppose that  $N$  is a special line bundle. For any  $u \in \text{Ext}^1(L, N)$ , consider the coboundary map induced by cohomology in the exact sequence (u):

$$\partial_u : H^0(L) \rightarrow H^1(N).$$

By exactness in (u) we have that  $h^1(C, E_u) = h^1(L) + \dim(\text{Coker}(\partial_u))$ . For any integer  $t > 0$ , consider the degeneracy locus

$$\mathcal{W}_t := \{u \in \text{Ext}^1(L, N) : \dim(\text{Coker}(\partial_u)) \geq t\},$$

which has a natural description of determinantal scheme. If  $m > 0$  and  $\mathcal{W}_t \neq \emptyset$ , any irreducible component  $\Delta_t \subseteq \mathcal{W}_t$  is such that  $\dim(\Delta_t) \geq \min\{m, m - t(t + h^0(L) - h^1(N))\}$ , where the right-hand-side is the expected dimension (cf. [5], Section 5.2). For the particular case of  $B^4(2, K_C)$ , if one tries to construct stable vector bundles with canonical determinant by extensions with  $h^1(C, N) = 0$  (so we are forced to have  $h^0(C, L) = 4$ ), we can get non-emptiness of  $B^4(2, K_C)$  but we cannot describe the general element of a component, so the condition on  $N$  to be a special line bundle give us an easier way to study this problem. Our main result is the following:

**Theorem.** Let  $C$  be a curve of genus  $g \geq 8$  with general moduli. Then  $B^4(2, K_C) \neq \emptyset$ . Moreover, there exists an irreducible component  $\mathcal{B} \subseteq B^4(2, K_C)$  of the expected dimension  $\rho_{K_C}(2, 4, g) = 3g - 13$  and whose general point  $[E]$  fits into an exact sequence

$$0 \rightarrow \mathcal{O}_C(D) \rightarrow E \rightarrow K_C(-D) \rightarrow 0,$$

where,

- (i).-  $D$  is a general effective divisor of degree 3, and
- (ii).-  $E = E_u$  with  $u \in \Delta_3 \subseteq \mathcal{W}_3 \subset \text{Ext}^1(K_C(-D), \mathcal{O}_C(D))$  general in  $\Delta_3$ , where  $\Delta_3$  is an irreducible component of dimension  $3g - 15$ , whose general element  $u \in \Delta_3$  satisfies  $\dim(\text{Cokernel}(\partial_u)) = 3$ .

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## 2. PRELIMINARIES.

We remind some definitions and results which are used for the proof of our main theorem. For more details on the topics of this section we refer the reader to e.g. ([1], [5], [8], [17]).

**2.1. Ruled surfaces and sections.** Let  $E$  be a vector bundle over a smooth irreducible complex projective curve  $C$  of genus  $g$ . The speciality of  $E$  is defined as  $i(E) = h^1(C, E)$ .  $E$  is said special if  $i(E) > 0$ .

If the rank of  $E$  is two, let  $S := \mathbb{P}(E)$  be the (geometrically) ruled surface associated to it, with structure map  $p : S \rightarrow C$ . For any  $x \in C$  we denote by  $f_x = p^{-1}(x) \simeq \mathbb{P}^1$ . We denote by  $f$  a general fiber of  $p$  and by  $\mathcal{O}_S(1)$  the tautological line bundle on  $S$ . We write  $d = \deg(E) = \deg(\wedge^2 E) = \deg(\det(E))$ . There is a map  $s : C \rightarrow S$  such that  $p \circ s = \text{Id}_C$  whose image we denote by  $H$  and such that  $\mathcal{O}_S(H) = \mathcal{O}_S(1)$ . Assume that  $h^0(C, E) > 0$ ; thus an element in the linear system  $|\mathcal{O}_S(1)|$  is denoted by  $H$ . For any  $D \in \text{Div}(C)$  ( or in  $\text{Pic}(C)$  ) we put  $f_D := p^*(D)$ . If  $\Gamma$  is a divisor on  $S$ , we set  $\deg(\Gamma) := H \cdot \Gamma$ . We have that  $d = \deg(E) = H^2 = \deg(H)$ . We recall that  $\text{Pic}(S) \simeq \mathbb{Z}[\mathcal{O}_S(1)] \oplus p^*(\text{Pic}(C))$ , moreover  $\text{Num}(S) \simeq \mathbb{Z} \oplus \mathbb{Z}$  and is generated by the classes  $H, f$  satisfying  $H \cdot f = 1, f^2 = 0$  (cf. [8], chapter V). Any element of  $\text{Pic}(S)$  corresponds to a divisor on  $S$  of the form  $nH + f_B, n \in \mathbb{Z}, B \in \text{Div}(C)$ , as an element of  $\text{Num}(F)$ , corresponds to  $nH + bf, b = \deg(B)$ . For any  $n \geq 0$  and for any  $D \in \text{Div}(C)$ , the linear system  $|nH + f_D|$  if non-empty, is said to be *n-secant* to the fibration  $p : S \rightarrow C$  since its general elements meets  $f$  at  $n$  points. An element  $\Gamma \in |H + f_D|$  is called *unisecant curve* of  $S$  (or to the fibration  $p : S \rightarrow C$ ). The irreducible unisecants of  $S$  are smooth and isomorphic to  $C$  and are called *sections* of  $S$ . We denote by  $\sim$  linear equivalence of divisors and by  $\equiv$  numerical equivalence of divisors.

We recall that there is a one-to-one correspondence between sections  $\Gamma$  of  $S$  and surjective maps  $E \twoheadrightarrow L$  with  $L$  a line bundle on  $C$  (cf. [8], chapter V), then one has an exact sequence  $0 \rightarrow N \rightarrow E \rightarrow L \rightarrow 0$ , where  $N \in \text{Pic}(C)$ . The surjection  $E \twoheadrightarrow L$  induces an inclusion  $\Gamma = \mathbb{P}(L) \subset S = \mathbb{P}(E)$ . If  $L = \mathcal{O}_C(B), B \in \text{Div}(C)$  with  $b = \deg(B)$ , then  $b = H \cdot \Gamma$  and  $\Gamma \sim H + f_N$  where  $N = L \otimes \det(E)^\vee \in \text{Pic}(C)$ . For a section  $\Gamma$  that corresponds to the exact sequence  $0 \rightarrow N \rightarrow E \rightarrow L \rightarrow 0$ , we have that  $\mathcal{N}_{\Gamma/S}$ , the normal sheaf of  $\Gamma$  in  $S$ , is such that  $\mathcal{N}_{\Gamma/S} = \mathcal{O}_\Gamma(\Gamma) \simeq N^\vee \otimes L$ . In particular  $\deg(\Gamma) = \Gamma^2 = \deg(L) - \deg(N)$ . If  $\Gamma \sim H - f_D$ , i.e. when  $N = \mathcal{O}_C(D)$ , then  $|\mathcal{O}_S(\Gamma)| \simeq \mathbb{P}(H^0(C, E(-D)))$ . For  $\Gamma \in \text{Div}(S)$ , we define  $\mathcal{O}_\Gamma(1) := \mathcal{O}_S(1) \otimes \mathcal{O}_\Gamma$ . The *speciality* of  $\Gamma$  is defined as  $i(\Gamma) := h^1(\Gamma, \mathcal{O}_\Gamma(1))$ .  $\Gamma$  is said *special* if  $i(\Gamma) > 0$ .

**2.2. Hilbert scheme of unisecant curves and Quot-Scheme.** For any  $n \in \mathbb{N}$ , denote by  $\text{Div}^{1,n}(S)$  the Hilbert scheme of unisecant curves of  $S$  which are of degree  $n$  with respect to  $\mathcal{O}_S(1)$ . Since elements of  $\text{Div}^{1,n}(S)$  correspond to quotients of  $E$ , therefore  $\text{Div}^{1,n}(S)$  can be endowed with a natural structure of Quot-Scheme (cf. [17], section 4.4), and one has an isomorphism (see e.g. [5], Section 2.4.)

$$\Phi_{1,n} : \text{Div}^{1,n}(S) \xrightarrow{\simeq} \text{Quot}_{E,n+a-g+1}^C, \quad \Gamma \rightarrow \{E \twoheadrightarrow L \oplus \mathcal{O}_A\},$$

where  $\Gamma$  is the unisecant  $\Gamma' + f_A$  with  $A \in \text{Div}(C)$ , and  $\Gamma'$  is the section corresponding to  $E \rightarrowtail L$ . The morphism  $\Phi_{1,n}$  allows to identify tangent spaces  $H^0(\Gamma, \mathcal{N}_{\Gamma/S}) \simeq T_{[\Gamma]}(\text{Div}^{1,n}(S)) \simeq \text{Hom}(N \otimes \mathcal{O}_C(-A), L \oplus \mathcal{O}_A)$  and obstruction spaces  $H^1(\Gamma, \mathcal{N}_{\Gamma/S}) \simeq \text{Ext}^1(N \otimes \mathcal{O}_C(-A), L \oplus \mathcal{O}_A)$  (cf. [17], Section 4.4).

For any integer  $m$  such that  $m \leq \overline{m} := \lfloor \frac{d-g+1}{2} \rfloor$ , one can consider also the set  $Q_m(E) := \{N \subset E : N \text{ is an invertible subsheaf of } E, \deg(N) = m\}$ . This set has a natural structure of Quot-scheme. Let  $\Gamma$  be any unisecant curve on  $S$  of degree  $n$ , corresponding to the exact sequence  $0 \rightarrow N \rightarrow E \rightarrow L \oplus \mathcal{O}_A \rightarrow 0$ , where  $L, N$  are line bundles such that  $n = \deg(L) + \deg(A)$  and  $m := \deg(N) = d - n$ . One has an isomorphism

$$\text{Div}^{1,n}(S) \xrightarrow{\Phi_{n,m}} Q_m(E), \quad \Phi_{n,m}([\Gamma]) = [N].$$

Moreover, one has a natural morphism  $\pi_m : Q_m(E) \rightarrow \text{Pic}^m(C), N \rightarrow [N]$ , and we denote  $W_m(E) := \text{Im}(\pi_m)$ . For any  $[N] \in W_m(E)$ , one has that  $\pi_m^{-1}([N]) \simeq \mathbb{P}(H^0(E \otimes N^\vee))$  (see [6], p. 199), this implies in particular that  $\pi_m$  has connected fibres. For every integer  $p \geq 0$ , one can consider the loci  $Q_m^p(E) := \{N \subset Q_m(E) : h^0(E \otimes N^\vee) \geq p+1\}$ ,  $W_m^p(E) := \{N \subset W_m(E) : h^0(E \otimes N^\vee) \geq p+1\}$ . For some properties of these loci we refer the reader to e.g. ([6]).

By the Quot-scheme structure of  $\text{Div}^{1,n}(S)$ , the universal quotient  $\mathcal{Q}_{1,\delta} \xrightarrow{\pi} \text{Div}^{1,n}(S)$  gives a morphism  $\text{Proj}(\mathcal{Q}_{1,\delta}) \xrightarrow{\pi} \text{Div}^{1,n}(S)$ . We define

$$\mathcal{S}^{1,n} := \{\Gamma \in \text{Div}^{1,n}(S) : R^1\pi_*(\mathcal{O}_{\mathcal{Q}_{1,\delta}}(1))_\Gamma \neq 0\}.$$

This locus supports the *scheme that parametrizes degree  $n$ , special unisecant curves of  $S$* .

**Definition 2.1.** ( cf. [5], Def. 2.2, 2.10) Let  $\Gamma \in \text{Div}^{1,n}(S)$  be. We say that:

- (i).-  $\Gamma$  is linearly isolated if  $\dim(|\mathcal{O}_S(\Gamma)|) = 0$ ;
- (ii).-  $\Gamma$  is algebraically isolated if  $\dim(\text{Div}^{1,n}(S)) = 0$ .

Let  $\Gamma$  be a special unisecant of  $S$ . Assume that  $\Gamma \in \mathcal{F}$ , where  $\mathcal{F} \subset \text{Div}^{1,n}(S)$  is a subscheme.

- (iii).-  $\Gamma$  is specially unique in  $\mathcal{F}$ , if  $\Gamma$  is the only special unisecant in  $\mathcal{F}$ .
- (iv).-  $\Gamma$  is specially isolated in  $\mathcal{F}$ , if  $\dim_\Gamma(\mathcal{F} \cap \mathcal{S}^{1,n}) = 0$ .

When  $\mathcal{F} = |\mathcal{O}_S(\Gamma)|$ ,  $\Gamma$  is said to be *linearly specially unique* in case (iii) and *linearly specially isolated* in case (iv).

When  $\mathcal{F} = \text{Div}^{1,n}(S)$ ,  $\Gamma$  is said to be *algebraically specially unique* in case (iii) and *algebraically specially isolated* in case (iv).

**2.3. A result on deformation theory.** Let  $Y$  be a smooth projective variety and  $j : X \subset Y$  be a closed smooth subvariety. Let  $\mathcal{I}_X \subset \mathcal{O}_Y$  be the ideal sheaf of  $X$ . We have an inclusion of tangent sheaves  $T_X \subset T_Y|_X$  and a restriction morphism

$T_Y \twoheadrightarrow T_Y|_X$ . Let  $T_Y \langle X \rangle \subset T_Y$  be the image inverse of  $T_X$  under the restriction morphism. The sheaf  $T_Y \langle X \rangle$  is called the *sheaf of germs of tangent vectors to  $Y$  which are tangents to  $X$* . This is a coherent sheaf of rank  $\dim(Y)$  on  $Y$ . We have a restriction map  $R : T_Y \langle X \rangle \twoheadrightarrow T_X$  giving the exact sequence  $0 \rightarrow T_Y(-X) \rightarrow T_Y \langle X \rangle \rightarrow T_X \rightarrow 0$ , where  $T_Y(-X)$  is the vector bundle of tangent vectors of  $Y$  vanishing along  $X$ . Let  $H^1(Y, T_Y \langle X \rangle) \xrightarrow{H^1(R)} H^1(X, T_X)$  be the map in cohomology induced by the above exact sequence. The map  $H^1(R)$  associates to a first-order deformation of  $(Y, X)$  the corresponding first-order deformation of  $X$ . We have (cf. [17], Proposition 3.4.17):

*First-order deformations of the pair  $(Y, X)$ .* The infinitesimal deformations of the pair  $(Y, X)$  (equivalently of the closed embedding  $j$ ) are controlled by the sheaf  $T_Y \langle X \rangle$ , that is,

- (i).- The obstructions lie in  $H^2(Y, T_Y \langle X \rangle)$ .
- (ii).- First-order deformations are parametrized by  $H^1(Y, T_Y \langle X \rangle)$  and the space  $H^0(Y, T_Y \langle X \rangle)$  parametrizes infinitesimal automorphisms.

**2.4. Extensions of line bundles and the Segre invariant.** Here we follow ([5] Sections 2.1 and 4.1), to which we refer the reader for more details.

For a rank two vector bundle  $E$ , the *Segre invariant*  $s(E)$  of  $E$  is defined as  $s(E) := \deg(E) - 2(\max\{\deg(N)\})$ , where the maximum is taken among all sub-line bundles  $N$  of  $E$ . The bundle  $E$  is stable (resp. semi-stable) if  $s(E) > 0$  (resp.  $s(E) \geq 0$ ). For any  $A \in \text{Pic}(C)$ , one has  $s(E) = s(E \otimes A)$ .

For  $\delta \leq d$  positive integers, take  $L \in \text{Pic}^\delta(C)$  special and effective, whereas  $N \in \text{Pic}^{d-\delta}(C)$  arbitrary. As in the Introduction, any  $u \in \text{Ext}^1(L, N)$  gives rise to an exact sequence  $(u) : 0 \rightarrow N \rightarrow E = E_u \rightarrow L \rightarrow 0$ , which therefore corresponds to a point in  $\mathbb{P} := \mathbb{P}(\text{Ext}^1(L, N))$ . Note that the bundle  $\mathcal{E}_e := E \otimes N^\vee$  fits into an exact sequence  $(e) : 0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{E}_e \rightarrow K_C \otimes N^\vee \rightarrow 0$ .

When  $\deg(L \otimes N^\vee) \geq 2$ ,  $\dim(\mathbb{P}) \geq 3$ , then the map  $\phi : C \rightarrow \mathbb{P}$  induced by  $|K_C \otimes L \otimes N^\vee|$  is a morphism and posing  $X := \phi(C) \subset \mathbb{P}$ , for any positive integer  $h$  one can consider  $\text{Sec}^h(X)$ , the  $h^{\text{th}}$ -secant variety of  $X$ , defined as the closure of the union of all linear spaces  $\langle \phi(D) \rangle \subset \mathbb{P}$  for general  $D \in \text{Sym}^{(h)}(C)$ . One has the following

**Proposition 2.1.** *Let  $2\delta - d \geq 2$ . For any integer  $\sigma$  such that  $\sigma \equiv 2\delta - d \pmod{2}$  and  $4 + d - 2\delta \leq \sigma \leq 2\delta - d$ , one has*

$$s(\mathcal{E}_e) \geq \sigma \Leftrightarrow e \notin \text{Sec}_{\frac{1}{2}(2\delta-d+\sigma-2)}(X).$$

See ([10]), Proposition 1.1.

The previous result gives in particular geometric conditions for  $E_u$  as above to be (semi)stable. When  $N$  is assumed to be special, as in [5], Section 5.2, one

can define for any positive integer  $t \geq 1$  the locus  $\mathcal{W}_t = \{u \in \text{Ext}^1(L, N) \mid \partial_u : H^0(L) \rightarrow H^1(N) \text{ has cokernel of dimension } \geq t\}$ , which has a natural structure of determinantal scheme, as such of expected dimension  $m - t(t + h^0(L) - h^1(N))$ , where  $m = \dim(\text{Ext}^1(L, N))$  as in Introduction.

**Definition 2.2.** (cf. [5], Def. 5.12). For  $h^0(L) \geq h^1(N) \geq t \geq 1$  integers, assume that:

(i).- There exists an irreducible component  $\Delta_t \subset \mathcal{W}_t$  of the expected dimension  $\dim(\Delta_t) = m - t(t + h^0(L) - h^1(N))$ ;

(ii).- For  $u \in \Delta_t$  general,  $\dim(\text{Coker}(\partial_u)) = t$ .

Any such  $\Delta_t$  is called a Good component of  $\mathcal{W}_t$ . See ([5], definition 5.12).

Theorems 5.8, 5.17 in ([5]) give sufficient conditions for the existence of good components. These have been given by the use of suitable multiplication maps and an alternative description of  $\mathcal{W}_t$ . Precisely, for any subspace  $W \subset H^0(K_C \otimes N^\vee)$  of dimension at least  $t$ , which turns out to be  $W = (\text{Coker}(\partial_u))^\vee$ , one considers the natural multiplication map of sections  $\mu : H^0(K_C \otimes N^\vee) \otimes H^0(L) \rightarrow H^0(K_C \otimes L \otimes N^\vee)$ . Let

$$\mu_W : W \otimes H^0(L) \rightarrow H^0(K_C \otimes L \otimes N^\vee) \quad (+)$$

be the restriction of  $\mu$  to  $W$ ; then  $\mathcal{W}_t$  is (cf. [5], Remark 5.7 for details):

$$\mathcal{W}_t = \{u \in H^0(K_C \otimes L \otimes N^\vee)^\vee : \exists W \subset H^0(K_C \otimes N^\vee) \text{ such that } \dim(W) \geq t \text{ and } \text{Im}(\mu_W) \subset \{u = 0\}\}. \quad (\star)$$

This is the description of  $\mathcal{W}_t$  we will use later on (cf. Lemma 3.2 and Step 1 of the proof of the main theorem).

**2.5. Global space of extensions.** For the general case of the following construction we refer to e.g ([1], [5] Section 6.).

Let  $C$  be a general curve of genus  $g \geq 3$ . Let  $0 \rightarrow N \rightarrow E \rightarrow L \rightarrow 0$  be an exact sequence such that  $\deg(N) = d - \delta > 0$ ,  $\deg(L) = \delta > 0$  and  $h^0(L) \cdot h^1(L) > 0$ ,  $h^0(N) \cdot h^1(N) > 0$ . Let  $\ell := h^0(L)$ ,  $j := h^1(L)$ ,  $n := h^0(N)$ ,  $r := h^1(N)$ . Set  $\mathcal{Y} := \text{Pic}^{d-\delta}(C) \times W_\delta^{\ell-1}(C)$ ,  $Z := W_{d-\delta}^{n-1}(C) \times W_\delta^{\ell-1}(C) \subset \mathcal{Y}$ . One has  $\dim(\mathcal{Y}) = g + \rho_L$  and  $\dim(Z) = \rho_L + \rho_N$ , where for any line bundle  $M$  we denote by  $\rho_M$  (or simply  $\rho$ , when  $M$  is understood) the usual Brill-Noether number.

Since  $C$  is general, we moreover have that  $\mathcal{Y}$  and  $Z$  are irreducible when  $\rho > 0$ ; otherwise one replace the (reducible) zero-dimensional Brill-Noether locus with one of its irreducible component to construct  $Y$  and  $Z$  as above.

When  $2\delta - d \geq 1$  one can construct a vector bundle  $\mathcal{E} \rightarrow \mathcal{Y}$  of rank  $m = 2\delta - d + g - 1$  (see [1], p. 176-180, [5] Section 6.), together with a projective bundle morphism  $\gamma : \mathbb{P}(\mathcal{E}) \rightarrow \mathcal{Y}$  where, for  $y = (N, L) \in \mathcal{Y}$ , the fiber  $\gamma^{-1}(y) = \mathbb{P}(\text{Ext}^1(N, L)) = \mathbb{P}$ .

We have that  $\dim(\mathbb{P}(\mathcal{E})) = \dim(\mathcal{Y}) + m - 1$ , and  $\dim(\mathbb{P}(\mathcal{E})|_Z) = \dim(Z) + m - 1$ . Since (semi)stability is an open condition, for  $2\delta - d \geq 2$  there is an open, dense subset  $\mathbb{P}(\mathcal{E})^0 \subseteq \mathbb{P}(\mathcal{E})$  and a morphism  $\pi_{d,\delta} : \mathbb{P}(\mathcal{E})^0 \rightarrow U_C(d)$ .

In ([5], Sections 6,7) the authors study the image and fibers of the map  $\pi_{d,\delta}$  under certain numerical conditions on  $d, \delta$  and in a more general context with respect to the line bundles  $L$  and  $N$ . They give in a different way a proof on the existence of irreducible and regular components in Brill-Noether loci  $B^j(r, d)$  and  $B^k(2, K_C), k \leq 3$ . Some of such components are the (dominant) image under  $\pi_{d,\delta}$  of certain degeneracy loci in  $\mathbb{P}(\mathcal{E})$  satisfying similar conditions to the good components of Definition 2.2. Such loci are called by the authors *Total good components* (see [5], Definition 6.13). Following this approach on a general curve  $C$  of genus  $g \geq 8$ , for  $k = 4$  we obtain an irreducible (not good) component  $\mathbb{P}(\widetilde{\Delta}_3) \subset \mathbb{P}(\mathcal{E})$  of dimension  $3g - 13$  that satisfies the conditions of the main theorem and fill-up an irreducible component  $\mathcal{B} \subseteq B^4(2, K_C)$  of dimension  $3g - 13$ . We prove this in the next section.

### 3. PROOF OF THE THEOREM.

For the following Lemma we adapt an argument of Lazarsfeld (cf. [7], Theorem 1.1).

**Lemma 3.1.** *Let  $C$  be a non-hyperelliptic curve of genus  $g \geq 8$  and let  $D = q_1 + q_2 + q_3$  be a general effective divisor of degree 3 on  $C$ . There exists a rank two vector bundle  $F$  on  $C$  with the following properties:*

- (i).-  $\det(F) = K_C(-D)$ ,  $h^0(F) = 3$  and  $F$  is globally generated,
- (ii).-  $h^0(F^\vee) = 0$ .

*Proof.* (i).- Consider  $p_1, \dots, p_{g-5}$  general points on  $C$ . Note that the line bundle  $A := K_C(-p_1 - \dots - p_{g-5})$  is of degree  $g + 3$ , base-point-free with  $h^0(C, A) = 5$ .  $M_1 := K_C \otimes A^\vee$  is a line bundle with only one section and  $h^0(C, M_1(-q_i)) = 0, i = 1, 2, 3$ . The line bundle  $M_2 := A(-D)$  is such that  $|M_2| = g_g^1$  is base-point-free. Consider the extension map  $\text{Ext}^1(M_2, M_1) \xrightarrow{\beta} \text{Hom}(H^0(M_2), H^1(M_1))$  which sends an extension

$$((e) : 0 \rightarrow M_1 \rightarrow E \rightarrow M_2 \rightarrow 0)$$

to the coboundary map

$$(\partial_e : H^0(M_2) \rightarrow H^1(M_1) = H^0(A)^\vee).$$

Any non-trivial extension in  $\text{Ker}(\beta)$  corresponds to a bundle  $F$  satisfying  $h^0(F) = 3$  and  $\det(F) = \wedge^2 F = K_C(-D)$ . We have the isomorphisms  $\text{Ext}^1(M_2, M_1) \simeq H^1(M_1 \otimes M_2^\vee) \simeq H^0(A^{\otimes 2}(-D))^\vee$  and  $\text{Hom}(H^0(M_2), H^1(M_1)) \simeq H^0(M_2)^\vee \otimes H^0(A)^\vee$ .



We prove that  $\text{Ker}(\beta)$  is a vector space of dimension  $g - 5$ : the map  $\beta$  is dual to the multiplication map  $m_D : H^0(M_2) \otimes H^0(A) \rightarrow H^0(M_2 \otimes A) = H^0(A^{\otimes 2}(-D))$ . By the base point free-pencil trick applied to  $M_2$ ,  $\text{Ker}(m_D) = H^0(C, \mathcal{O}_C(D))$ , then  $\dim \text{Ker}(m_D) = 1$  and  $m_D$  has cokernel of dimension  $h^0(A^{\otimes 2}(-D)) - 9 = g - 5 = \dim \text{Ker}(\beta)$ .

Now consider an extension  $((e) : 0 \rightarrow M_1 \rightarrow F \rightarrow M_2 \rightarrow 0)$  in  $\text{Ker}(\beta)$  and suppose that  $F$  is not globally generated, then the three sections of  $F$  generates a subsheaf  $F_1$  of  $F$  fitting into an exact sequence  $0 \rightarrow M_1(-B) \rightarrow F_1 \rightarrow M_2 \rightarrow 0$ , for some divisor  $B$  over  $C$ . For any  $y \in C - \{p_1, \dots, p_{g-5}\}$ ,  $h^0(M_1(-y)) = 0$  and  $h^0(M_2(-y)) = 1$ , then  $h^0(F(-y)) \leq 1$ . By Riemann-Roch we have that  $h^0(F) - h^0(F^\vee \otimes K_C) = -3$ , that is,  $h^0(K_C \otimes F^\vee) = 6$ . This implies that  $h^0(F(-y)) = -5 + h^0(K_C(y) \otimes F^\vee) \geq 1$ , so  $h^0(F(-y)) = 1$  and  $F$  is globally generated at  $C - \{p_1, \dots, p_{g-5}\}$ . This implies that  $B \subset \text{supp}(\{p_1, \dots, p_{g-5}\})$ . Thus, if  $F$  comes from an element  $e \in \text{Ker}(\beta)$  that fails to be generated by global sections, then there is a point  $x$  among the points  $\{p_1, \dots, p_{g-5}\}$  and a subsheaf  $F_2$  given by the extension  $0 \rightarrow M_1(-x) \rightarrow F_2 \rightarrow M_2 \rightarrow 0$  so that the extension  $(e)$  is induced from this latter extension and it is surjective on global sections. Since  $\text{Hom}(H^0(M_2), H^1(M_1(-x))) \simeq \text{Hom}(H^0(M_2), H^0(A(x)))$ , such extensions are parametrized by elements in

$$\text{Ker} [\text{Ext}^1(M_2, M_1(-x)) \rightarrow \text{Hom}(H^0(M_2), H^0(A(x)))],$$

where  $\text{Ext}^1(M_2, M_1(-x)) \simeq H^0(A^{\otimes 2}(x - D))^\vee$ . Note that since  $h^0(A) = 5$  and  $x \in \text{supp}(\{p_j\}_{j=1}^{g-5})$  then  $h^0(A(x)) = 6$ . By the base point free pencil trick applied to  $M_2$ , note that  $H^0(C, \mathcal{O}_C(x + D))$  is the kernel of the map  $H^0(M_2) \otimes H^0(A(x)) \rightarrow H^0(A^{\otimes 2}(x - D))$  and  $h^0(C; \mathcal{O}_C(x + D)) = 1$ , then the cokernel has dimension  $h^0(A^{\otimes 2}(x - D)) - 11 = g - 6$ . This implies that the extensions in  $\text{Ker}(\beta)$  which fail to be generated by global sections have codimension at least 1 in  $\text{Ker}(\beta)$ , so for a general extension in  $\text{Ker}(\beta)$ , the corresponding vector bundle  $F$  satisfies that  $F$  is globally generated,  $h^0(F) = 3$  and  $\wedge^2 F = K_C(-D)$ .

(ii).- We have the identification  $F \simeq F^\vee \otimes K_C(-D)$ . Since  $F$  is globally generated, one has:

$$0 \rightarrow (K_C(D))^\vee \rightarrow H^0(C, F) \otimes \mathcal{O}_C \rightarrow F \rightarrow 0. \quad (1)$$

Take  $V = H^0(C, F)^\vee$  and dualize (1) to get

$$0 \rightarrow F^\vee \rightarrow V \otimes \mathcal{O}_C \rightarrow K_C(-D) \rightarrow 0, \quad (2)$$

thus  $h^0(F^\vee \otimes K_C(-D)) = h^0(F) = 3$ . If  $0 \neq s \in H^0(F^\vee)$ , then we have an injection  $s : \mathcal{O}_C \hookrightarrow F^\vee$ , so  $K_C(-D) \hookrightarrow F^\vee \otimes K_C(-D) = F$  and  $3 = h^0(F) \geq h^0(K_C(-D)) = g - 3$  which is a contradiction for  $g \geq 7$ , then  $h^0(F^\vee) = 0$ .  $\square$

**Lemma 3.2.** *Let  $\mathbb{G} := G(3, H^0(C, K_C(-D)))$  be the Grassmannian of 3-planes in  $H^0(C, K_C(-D))$ . For  $V \in \mathbb{G}$  general, the map*

$$\mu_V : V \otimes H^0(C, K_C(-D)) \rightarrow H^0((K_C(-D))^{\otimes 2})$$

*as in (+) has kernel of dimension 3.*

*Proof.* Note that  $\wedge^2 V \subset \text{Ker}(\mu_V)$ , then  $\dim \text{Ker}(\mu_V) \geq 3, \forall V \in \mathbb{G}$ . Let  $\Sigma_3 := \{V \in \mathbb{G} : \dim \text{Ker}(\mu_V) = 3\}$ . We are going to show that  $\Sigma_3 \neq \emptyset$  and that  $\dim(\Sigma_3) = \dim(\mathbb{G})$ .

Consider the vector bundle  $F$  constructed in Lemma 3.1. Tensoring (2) by  $K_C(-D)$  we have

$$0 \rightarrow F^\vee \otimes K_C(-D) \rightarrow H^0(F)^\vee \otimes K_C(-D) \rightarrow (K_C(-D))^{\otimes 2} \rightarrow 0. \quad (3)$$

We know that  $h^0(F^\vee) = 0$ , also a non-zero section  $\tau \in H^0(\mathcal{O}_C)$  induces an isomorphism  $H^0(F)^\vee \otimes H^0(\mathcal{O}_C) \simeq H^0(F)^\vee$ , hence taking cohomology in (2) we have an injective map  $\iota : H^0(F)^\vee \hookrightarrow H^0(K_C(-D))$ . Set  $V := \iota(H^0(F)^\vee)$ , then  $V \subset H^0(K_C(-D))$  has dimension 3. Since  $F^\vee \otimes \det(F) \simeq F$ , then  $H^0(F^\vee \otimes K_C(-D)) \simeq H^0(F)$ , so we take cohomology in (3) to obtain

$$0 \rightarrow H^0(C, F) \rightarrow H^0(F)^\vee \otimes H^0(K_C(-D)) \rightarrow H^0((K_C(-D))^{\otimes 2}) \rightarrow \dots \quad (4)$$

From (4) we have that

$$H^0(C, F) \simeq \text{Kernel}(H^0(F)^\vee \otimes H^0(K_C(-D)) \rightarrow H^0((K_C(-D))^{\otimes 2})). \quad (5)$$

Since  $H^0(F)^\vee \otimes H^0(K_C(-D)) \simeq V \otimes H^0(K_C(-D))$ , from (5) we have  $H^0(F) \simeq \text{Ker}(\mu_V)$ , then  $V \simeq H^0(C, F)$ , so  $\Sigma_3 \neq \emptyset$ . By upper semicontinuity of the function  $\mathbb{G} \rightarrow \mathbb{Z}, V \rightarrow \dim \text{Ker}(\mu_V)$ , we have that for the general  $V \in \mathbb{G}$ ,  $\dim \text{Ker}(\mu_V) = 3$ , then  $\dim(\Sigma_3) = \dim(\mathbb{G})$ .  $\square$

### Proof of the Theorem:

**Step 1.-** Here we use the same strategy as in proofs of Theorems 5.8 and 5.17 in [5]. Let  $\mathbb{P} = \mathbb{P}((H^0(K_C(-D))^{\otimes 2})^\vee) \simeq \mathbb{P}(\text{Ext}^1(K_C(-D), \mathcal{O}_C(D)))$ . By Lemma 3.2 and the description of  $\mathcal{W}_i$  as in  $(\star)$  we have that  $\emptyset \neq \mathcal{W}_3 \subset \mathbb{P}$  so it has expected dimension  $3g - 18$ . Denote by  $\pi_u$  the hyperplane in  $\mathbb{P}$  defined by  $\{u = 0\} \subset H^0((K_C(-D))^{\otimes 2})$ , where  $u$  corresponds to the extension  $(u) : 0 \rightarrow \mathcal{O}_C(D) \rightarrow E \rightarrow K_C(-D) \rightarrow 0$ .

Let  $\mathcal{J}_{\mathbb{G}} := \{(W, \pi) \in \mathbb{G} \times \mathbb{P} : \text{Im}(\mu_W) \subset \pi\}$  be and let  $\pi_1 : \mathcal{J}_{\mathbb{G}} \rightarrow \mathbb{G}, \pi_2 : \mathcal{J}_{\mathbb{G}} \rightarrow \mathbb{P}$  be the projections to the first and second factor respectively. From Lemma 3.2 we have that for  $W \in \mathbb{G}$  general,  $\dim(\text{Im}(\mu_W)) = 3g - 12$ . The fiber of  $\pi_1$  over a general element  $V \in \Sigma_3$ , is isomorphic to the linear system of hyperplanes in  $\mathbb{P}$  passing through the linear subspace  $\mathbb{P}(\text{Im}(\mu_V))$ , then the general fiber of  $\pi_1$  is irreducible and of dimension  $(h^0((K_C(-D))^2) - 1) - (3g - 12) = 2$ . On the other hand, since  $\Sigma_3$

is dense in  $\mathbb{G}$  and  $\Sigma_3 \subseteq \pi_1(\mathcal{J}_{\mathbb{G}})$ , then  $\mathcal{J}_{\mathbb{G}}$  dominates  $\mathbb{G}$  through  $\pi_1$ , thus there exists a unique component  $\mathcal{J}_3$  dominating  $\mathbb{G}$  through  $\pi_1$  and  $\dim(\mathcal{J}_3) = 2 + \dim(\mathbb{G}) = 2 + 3(g - 6) = 3g - 16$ . By Serre duality,  $\partial_u : H^0(C, K_C(-D)) \rightarrow H^1(\mathcal{O}_C(D))$  is symmetric, that is,  $\partial_u = \partial_u^\vee$ , then  $\text{Ker}(\partial_u) = \text{Ker}(\partial_u^\vee) = (\text{Im}(\partial_u))^\perp$ , in particular  $\text{Ker}(\partial_u)$  is uniquely determined, so the general fiber of the map  $\pi_2|_{\mathcal{J}_3} : \mathcal{J}_3 \rightarrow \mathbb{P}$  is irreducible and zero-dimensional, then  $\overline{\pi_2(\mathcal{J}_3)} \subset \mathbb{P}(\mathcal{W}_3) \subset \mathbb{P}$  is irreducible of dimension  $3g - 16$ . Note that  $\overline{\pi_2(\mathcal{J}_3)}$  gives rise to the existence of a component  $\Delta_3 \subset \mathcal{W}_3$  of dimension  $3g - 15$ , which is therefore not good (in the sense of Definition 2.2) with  $\mathbb{P}(\Delta_3) = \overline{\pi_2(\mathcal{J}_3)}$ , such that for the general element  $u \in \Delta_3$  we have that  $\partial_u : H^0(C, K_C(-D)) \rightarrow H^1(C, \mathcal{O}_C(D))$  has cokernel of dimension 3.

Consider the image  $X$  of the map  $C \hookrightarrow \mathbb{P}$  defined by the (very ample) linear system  $|(K_C(-D))^{\otimes 2}|$ . Note that for  $\epsilon \in \{0, 1\}$  and for  $\sigma = g - 6 - \epsilon > 0$  we have that  $3g - 16 = \dim(\mathbb{P}(\Delta_3)) > \dim \text{Sec}_{\frac{1}{2}(3g-16-\epsilon)}(X)$ , then by Proposition 2.1 we have that for general  $u \in \Delta_3$ , the associated bundle  $E_u$  is stable. This shows in particular that  $B^4(C, K_C) \neq \emptyset$  and describes some points in it.

**Step 2.- Unobstructed sections.** Let  $u \in \Delta_3$  be a general extension and let  $E = E_u$  be the associated vector bundle which fits into the exact sequence  $(u) : 0 \rightarrow \mathcal{O}_C(D) \rightarrow E \rightarrow K_C(-D) \rightarrow 0$ . Let  $\Gamma$  be the section corresponding to the quotient  $E \twoheadrightarrow K_C(-D)$ . Let  $S = \mathbb{P}(E)$  and  $p : S \rightarrow C$  be the structure map. Let  $c$  be the class of  $\mathcal{O}_S(1)$  in  $\text{Pic}(S)$  (or  $\text{Num}(S) \simeq H^2(S, \mathbb{Z})$ ). The tangent bundle  $T_S$  fits in an exact sequence

$$0 \rightarrow T_{S/C} \rightarrow T_S \rightarrow p^*T_C \rightarrow 0, \quad (6)$$

where  $T_{S/C} := \text{Ker}(T_S \rightarrow p^*(T_C))$  is the relative tangent sheaf (the sheaf of tangent vectors along the fibers of  $p$ ). The sheaf  $T_{S/C}$  is dual to the relative canonical sheaf  $\omega_{S/C}$  and  $\omega_{S/C} = \mathcal{O}_S(-2c) \otimes p^*(\det(E)) = \mathcal{O}_S(-2c) \otimes p^*(K_C)$ , then  $T_{S/C} = \mathcal{O}_S(2c) \otimes p^*(T_C)$ . On the other hand we have that  $\mathcal{N}_{\Gamma/S} \simeq K_C(-2D)$ , then  $h^0(\Gamma, \mathcal{N}_{\Gamma/S}) = g - 6$  and  $h^1(\Gamma, \mathcal{N}_{\Gamma/S}) = 1$ . By ([17], p. 177, eq. (3.56)) We have

$$0 \rightarrow T_S(-\Gamma) \rightarrow T_S < \Gamma > \rightarrow T_\Gamma \rightarrow 0. \quad (7)$$

Tensoring the exact sequence (6) by  $\mathcal{O}_S(-\Gamma)$  we have

$$0 \rightarrow T_{S/C}(-\Gamma) \rightarrow T_S(-\Gamma) \rightarrow p^*(T_C) \otimes \mathcal{O}_S(-\Gamma) \rightarrow 0. \quad (8)$$

Recall that  $\Gamma \sim \mathcal{O}_S(1) - f_D$  (see Section 2.1). Since  $K_S \equiv -2c + (4g - 4)f$ , then  $K_S(\Gamma) \equiv -c + a_0f$  for some  $a_0 \in \mathbb{Z}$ . From the isomorphism  $T_{S/C} \simeq \mathcal{O}_S(2c) \otimes p^*(T_C)$  we have that  $K_S(\Gamma) \otimes (T_{S/C})^\vee \equiv -3c + af$  for some integer  $a$ , in particular  $-3c + af$  is non-effective, then  $h^2(T_{S/C}(-\Gamma)) = h^0(K_S(\Gamma) \otimes (T_{S/C})^\vee) = 0$ . Similarly we have that  $K_S(\Gamma) - p^*(T_C) \equiv -c + bf$  for some integer  $b$ ; then it is non-effective, so  $h^2(p^*(T_C) \otimes \mathcal{O}_S(-\Gamma)) = h^0(K_S(\Gamma) \otimes (p^*(T_C))^\vee) = 0$ . From (8) we have that  $h^2(T_S(-\Gamma)) = 0$ , and from (7) we deduce that  $h^2(T_S < \Gamma >) = 0$ , then  $(\Gamma, S)$  is

unobstructed (see Section 2.3), that is, the first-order infinitesimal deformations of the closed embedding  $\Gamma \hookrightarrow S$  are unobstructed with  $S$  not fixed, in particular  $\Gamma$  is unobstructed in  $S$  fixed and  $\Gamma$  varies in a  $(g - 6)$ -dimensional family.

**Step 3.- Specially isolated sections.** Let

$$(u) : 0 \rightarrow \mathcal{O}_C(D) \rightarrow E = E_u \rightarrow K_C(-D) \rightarrow 0$$

be a general extension in  $\mathbb{P}(\Delta_3)$ . Consider the quotient  $E_u \twoheadrightarrow K_C(-D)$  and the corresponding section  $\Gamma_u = \Gamma$ . To show that  $\Gamma$  is specially isolated, we need to show that there are only finitely many sections corresponding to special quotients like these, that is, we need to show that the family of such corresponding quotients (which vary in a  $(g - 6)$ -dimensional family) does not intersect in positive dimension the 3-dimensional family  $\mathcal{F} = \{L_t\}_{t \in T \subset \text{Sym}^3(C)}$  of special line bundles. By construction, we can assume that the family  $\mathcal{F}$  is (isomorphic to)  $W_{2g-5}^{g-4}(C) \simeq W_3^0(C)$ .

Associated to the extension  $(u)$  we have an exact cohomology sequence

$$0 \rightarrow H^0(C, \mathcal{O}_C(D)) \rightarrow H^0(C, E) \xrightarrow{\alpha} H^0(C, K_C(-D)) \xrightarrow{\partial_u} H^1(C, \mathcal{O}_C(D)) \rightarrow \dots$$

where by Step 1,  $\text{Im}(\alpha) = \text{Ker}(\partial_u)$  is a general element in  $\mathbb{G}$ . Suppose that  $h^0(C, E(-D)) > 1$ , then there exists a non-zero section  $s \in H^0(C, E)$ ,  $s \notin H^0(C, \mathcal{O}_C(D))$  such that  $s$  vanishes at the points of  $D$ , then we have that  $\alpha(s) \in \text{Ker}(\partial_u) \subset H^0(C, K_C(-D))$  is a non-zero section that vanishes at the points of  $D$ , that is,  $\alpha(s) \in \text{Ker}(\partial_u) \cap H^0(C, K_C(-2D))$ , thus  $\text{Ker}(\partial_u)$  is an element of the proper and closed sublocus  $\{W \in \mathbb{G} : W \cap H^0(C, K_C(-2D)) \neq \emptyset\} \subsetneq \mathbb{G}$ , since  $\text{Ker}(\partial_u)$  is general in  $\mathbb{G}$  this contradiction shows that  $h^0(C, E(-D)) = 1$ , which implies by Section 2.1 that  $\dim|\mathcal{O}_S(\Gamma)| = 0$ , that is,  $\Gamma$  is linearly isolated (see Definition 2.1).

Let  $N = \mathcal{O}_C(D)$ ; by construction we have that  $[N]$  is a general point in  $W_3(E)$  (see Section 2.2). Since  $h^0(E(-D)) = 1$ , the fiber  $\pi_3^{-1}([N])$  of the map  $\pi_3 : Q_3(E) \rightarrow W_3(E)$  at  $[N]$  is only one point (see Section 2.2). Tensoring by  $N^\vee = \mathcal{O}_C(-D)$  the exact sequence induced by  $(u)$ , we have in cohomology the exact sequence

$$0 \rightarrow H^0(\mathcal{O}_C) \rightarrow H^0(E(-D)) \rightarrow H^0(K_C(-2D)) \xrightarrow{\delta} H^1(\mathcal{O}_C) \rightarrow \dots$$

where the map  $\delta$  can be identified with the differential of  $\pi_3 : Q_3(E) \rightarrow W_3(E)$ , then  $\delta$  is an isomorphism onto its image, that is,

$$H^0(K_C(-2D)) = T_{[N]}(Q_3(E)) \simeq T_{[N]}(W_3(E)) = \delta(H^0(K_C(-2D))) \subseteq H^1(C, \mathcal{O}_C).$$

Denote by  $\langle, \rangle$  the Serre duality pairing, and let  $V_1 := H^0(K_C(-2D))$ . With these identifications of tangent spaces, to prove that  $\Gamma$  is specially isolated we need to prove that  $T_{[N]}(W_3(E)) \cap T_{[N]}(W_3^0(C)) = (0) = \delta(V_1) \cap (\text{Im}(\mu_N))^\perp$ , where for  $j = 1, 2$ ,  $\mu_{N^j} : H^0(\mathcal{O}_C(jD)) \otimes H^0(K_C(-jD)) \rightarrow H^0(C, K_C)$  is the Petri map. Since  $\text{Im}(\mu_N) = H^0(K_C(-D)) \supset \text{Im}(\mu_{N^2}) = V_1$ , then  $(\text{Im}(\mu_N))^\perp \subset V_1^\perp$ . Let  $\delta(\omega) \in \delta(V_1) \cap (\text{Im}(\mu_N))^\perp \subset \delta(V_1) \cap V_1^\perp$ . Then  $\langle \delta(\omega), v \rangle = 0$  for all  $v \in V_1$ , that is,  $\delta(\omega) \in \text{Ker}(V_1 \xrightarrow{\beta} V_1^\vee)$ , where  $\beta : x \rightarrow \beta_x, \beta_x(v) = \langle x, v \rangle$ . By duality,  $\beta$  is

an isomorphism, then  $\delta(\omega) = 0$ ; since  $\delta$  is injective,  $\omega = 0$ . This prove that  $\Gamma$  is specially isolated.

**Step 4. The map  $\mathbb{P}(\Delta_3) \rightarrow B^4(2, K_C)$  is generically injective.** By what proved in Step 1 we can consider the map  $\pi : \mathbb{P}(\Delta_3) \rightarrow B^4(2, K_C)$

$$((u) : 0 \rightarrow \mathcal{O}_C(D) \rightarrow E_u \rightarrow K_C(-D) \rightarrow 0) \longrightarrow [E_u]. \quad (9)$$

For a general element  $[E_u] \in \pi(\mathbb{P}(\Delta_3)) \subset B^4(2, K_C)$ , we have that  $\pi^{-1}([E_u])$  corresponds to the extensions  $u' \in \mathbb{P}(\Delta_3)$  that induce exact sequences  $0 \rightarrow \mathcal{O}_C(D) \rightarrow E_{u'} \rightarrow K_C(-D) \rightarrow 0$  and a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_C(D) & \xrightarrow{\iota_1} & E_u & \longrightarrow & K_C(-D) \longrightarrow 0 \\ & & & & \downarrow \phi & & \\ 0 & \longrightarrow & \mathcal{O}_C(D) & \xrightarrow{\iota_2} & E_{u'} & \longrightarrow & K_C(-D) \longrightarrow 0 \end{array} \quad (10)$$

with an isomorphism  $\phi : E_u \rightarrow E_{u'}$  of stable bundles. The maps  $\phi \circ \iota_1$  and  $\iota_2$  determine two non-zero sections  $\sigma_1 \neq \sigma_2$  in  $H^0(E_u(-D))$ . For Step 3, we have that for the general extension  $u \in \Delta_3$ , the corresponding section  $\Gamma$  of the quotient line bundle  $E_u \rightarrow K_C(-D)$  is linearly isolated, then  $h^0(C, E_u(-D)) = 1$ , so there exists some scalar  $\lambda \neq 0$  such that  $\phi \circ \iota_1 = \lambda \iota_2$ . Since  $E_u$  is stable, by Lemma 4.5 in ([5]) we have that  $E_u = E_{u'}$ , i.e  $u, u'$  are proportional in  $\Delta_3$ , then  $\pi$  is generically injective. In particular for  $u, u' \in \Delta_3$  general points, the vector bundles  $E_u, E_{u'}$  cannot be isomorphic.

**Step 5. A global moduli map.** Given a special and effective line bundle  $L \in \text{Pic}(C)$  which is assumed to be a quotient line bundle of  $E$ , the condition for  $E$  to have canonical determinant forces the kernel  $N$  of the quotient map  $E \twoheadrightarrow L$  to be isomorphic to  $K_C \otimes L^\vee$ . For  $D$  a general effective divisor of degree three,  $L = K_C(-D)$  depends on  $\rho(L) := \rho(g, g-4, 2g-5) = 3$  parameters. From Steps 1-3 and the construction in ([5], Section 6) we obtain an irreducible component  $\mathbb{P}(\widetilde{\Delta}_3) \subset \mathbb{P}(\mathcal{E})$  of dimension  $3g-16+\rho(L) = 3g-13$ , where a point in  $\mathbb{P}(\widetilde{\Delta}_3)$  corresponds to the datum of a pair  $(y, u)$  with  $y = (\mathcal{O}_C(D), K_C(-D))$ ,  $D$  a general and effective divisor of degree 3 and  $u \in \mathbb{P}(\Delta_3)$  an extension as in Step 1. From Steps 3 and 4 we have that the global moduli map  $\pi_{2g-2, 2g-5}|_{\mathbb{P}(\widetilde{\Delta}_3)} : \mathbb{P}(\widetilde{\Delta}_3) \rightarrow B^4(2, K_C)$  is generically injective and its image fills up an irreducible component  $\mathcal{B} \subset B^4(2, K_C)$  of dimension  $\rho_{K_C}(2, 4, g) = 3g-13$ .  $\square$

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